Arrival and departure time in quantum mechanics

1984 J. Phys. A: Math. Gen. 17 1473

(http://iopscience.iop.org/0305-4470/17/7/014)

J. Phys. A: Math. Gen. 17 (1984) 1473–1478. Printed in Great Britain

Arrival and departure time in quantum mechanics

J E G Farina
Mathematics Department, University of Nottingham, Nottingham, NG7 2RD, UK

Received 19 October 1983, in final form 12 January 1984

Abstract. The arrival time at a point in the case of the one-dimensional motion of a classical particle is only predictable if initially the position and velocity are both known precisely. It is shown that such an arrival time can be defined in a probabilistic sense when only the initial means and standard deviations of position and velocity are known. The arrival time so defined depends on the subjective concept of confidence limit. It is further shown that arrival time in the latter sense goes over to quantum mechanics. A lower bound on the transit time is derived for this situation by use of the Mandelstam–Tamm inequality.

1. Introduction

The status of time in non-relativistic quantum mechanics has received extensive treatment, and no attempt will be made here to give even a partial review of the literature. Extensive discussion has taken place on delay time and the closely related topic of decay time, the treatment of time as an observable rather than as a parameter, and on the time-energy uncertainty principle. In this work we shall concentrate on a special and concrete situation, namely the arrival time at a point of a particle moving freely in one dimension. We shall discuss this concept both in the case when the particle is governed by the laws of classical mechanics, and also in the case when the particle is governed by the laws of quantum mechanics.

It is well known that an approach to arrival time as an observable conjugate to energy involves mathematical difficulties. Such difficulties, and others, were considered by Allcock in a series of three papers (Allcock 1969a, b, c). In order to meet them, he took into account, in a quantum mechanical way, the effect both of the source and of the measuring apparatus. Arrival time at the point $x = 0$ was then interpreted in terms of the probability $P(t)$ that at time t the particle should have arrived at, and been registered by, the apparatus, situated in the interval $x \geqslant 0$.

In his work Allcock has an apparatus present in the region $x \geqslant 0$ throughout time. As he points out it can be visualised as a succession of 'sweeps' of the positive real axis, and can be modelled by an absorbtive potential $-\mathrm{i}V$ in the interval $x \geqslant 0$. The 'watched pot' problem has also been examined by other workers (for example, Misra and Sudarshan 1977, Davies 1976).

Here we shall look at the problem from a different point of view, and so avoid the 'watched pot' difficulties. We shall consider a particle emitted by a source S, whose right-hand end has coordinate $x = s$, and which then travels along the line $0x$. It will be assumed that S is well to the left of the origin 0, so that at any time t when the particle is in the neighbourhood of 0 it is moving freely. The region $x \geqslant a$, where

0305-4470/84/071473+06\$02.25

$x = a > 0$ is the coordinate of a point A to the right of 0, may be 'instanteously swept' at any time. The problem then becomes: 'Is there a time t_1 before which the probability of the particle being found in the region $x \geqslant a$ is insignificant? Is there a second time t_2 after which the probability of the particle being found in the region $x \geqslant a$ is not significantly different from unity?' By the nature of the questions, if there is an answer to either then it is certainly not unique; for example if t_1 exists and $t_1' < t_1$ then t_1' is also such a time.

Let $P[x \geqslant a | t]$ be the probability that the particle is to the right of A at time t. Whether this is 'insignificant', 'significant', or 'insignificantly different from unity' is a subjective judgement by a human being. Such a judgement, described by statisticians as a 'confidence limit', takes the form:

'If $P[x \geqslant a | t] < \varepsilon$ the probability that the particle is to the right of A is insignificant. If $P[x \geqslant a | t] > 1 - \varepsilon$ the probability that the particle is to the left of A is insignificant.' For example, $\varepsilon = 0.1$ represents a 10% confidence limit. Given ε it then may be possible to determine t_1 and t_2 such that

$$t < t_1 \Rightarrow P[x \geqslant a | t] < \varepsilon$$
$$t > t_2 \Rightarrow P[x \geqslant a | t] > 1 - \varepsilon.$$

Then t_1 can be regarded as the arrival time, t_2 the departure time, and $t_2 - t_1$ the transit time, of the position probability distribution at the point A.

A comment here on the subjectivity, or otherwise, of the above concepts may be in order. An estimate of $P[x \geqslant a | t]$ is subjective in the sense that is is made by the experimentalist. However, the estimate is forced upon him by the objective conditions of the experiment. If he wishes to improve his estimate he must improve his apparatus, which entails altering the objectively existing conditions of the experiment. Even the estimate itself may not depend on a conscious observer—it could be calculated by a computer attached to the apparatus and displayed on a visual display unit. The choice of ε is, on the other hand, the result of a human decision. Given this choice, however, the consequences follow in a logical and objective way.

The above definitions of t_1 and t_2 are equally valid whether the particle obeys the laws of quantum mechanics or classical mechanics. In classical mechanics we have a probability measure $p(x, v)$ on phase space; in quantum mechanics a probability measure on the lattice of subspaces of a Hilbert space. In either case the probabilitiy measures evolve freely in the asymptotic limit $t \to +\infty$ (Simon and Reed 1979), when the effect of the source therefore vanishes.

In § 2 we shall show that given a confidence limit ε, and the means and standard deviations of position and velocity at time $t = 0$, there exists at any time t an interval $\mathscr{I}_t$ to which the particle is confined to within this confidence limit. This is used in § 3 to define arrival time, departure time and transit time, and determine conditions for their existence. The Mandelstam–Tamm inequality is used in § 4 to show that the transit time, when it exists, satisfies a time–energy uncertainty principle.

2. The interval $\mathscr{I}_t$

Let $q(x)$ be the probability density at $t = 0$ for a particle moving freely along the x axis. In the case of a classical particle $q(x) = \int p(x, v)\, \mathrm{d}v$. In the case of a quantum particle with wavefunction φ, $q(x) = |\varphi(x)|^2$; if the particle is in a mixed state q will be a convex linear combination of such terms. The mean position is $\langle x\rangle = \int xq(x)\,\mathrm{d}x$ and the standard deviation Δx is given by

$$(\Delta x)^2 = \int (x - \langle x\rangle)^2 q(x)\,\mathrm{d}x. \tag{2.1}$$

Let l be any positive number. Then (2.1) can be written

$$(\Delta x)^2 = \int_{\langle x\rangle - l}^{\langle x\rangle + l} (x - \langle x\rangle)^2 q(x)\,\mathrm{d}x + r \tag{2.2}$$

where

$$r = \left(\int_{-\infty}^{\langle x\rangle - l} + \int_{\langle x\rangle + l}^{+\infty}\right)(x - \langle x\rangle)^2 q(x)\,\mathrm{d}x. \tag{2.3}$$

In the integrand on the right-hand side of (2.3), $(x - \langle x\rangle)^2 \geqslant l^2$. Hence

$$r \geqslant l^2 P(x \not\in \mathcal{I}) \tag{2.4}$$

where

$$P(x \not\in \mathcal{I}) \equiv \left(\int_{-\infty}^{\langle x\rangle - l} + \int_{\langle x\rangle + l}^{+\infty}\right) q(x)\,\mathrm{d}x$$

is the probability that the particle lies outside the interval $\mathcal{I} \equiv [\langle x\rangle - l, \langle x\rangle + l]$. The first term on the right-hand side of (2.2) is non-negative and so using (2.4)

$$(\Delta x/l)^2 \geqslant P(x \not\in \mathcal{I}). \tag{2.5}$$

Thus if, for example, $l = 10\Delta x$ we have a 1% confidence limit that the particle is confined to the interval $\mathcal{I}$.

It can be shown (Farina 1977, 1982) that, for free motion, whether the particle is classical or quantum, the standard deviation Δx_t of position at time t is given by

$$(\Delta x_t)^2 = (\Delta x)^2 + 2\,\mathrm{cov}(x, v)t + (\Delta v)^2 t^2 \tag{2.6}$$

where Δv is the standard deviation of velocity, and $\mathrm{cov}(x, v)$ is the covariance of position and velocity, at time $t = 0$. For a classical particle this latter is $\langle (x - \langle x\rangle)(v - \langle v\rangle)\rangle$ where $\langle f\rangle = \iint f(x, v)p(x, v)\,\mathrm{d}x\,\mathrm{d}v$ is the expectation value of a random variable f on phase space at time $t = 0$. For a quantum particle the appropriate definition is $\mathrm{cov}(x, v) = \langle (x - \langle x\rangle) \circ (v - \langle v\rangle)\rangle$, the symmetrised product $A \circ B$ of two operators A and B being defined as $\frac{1}{2}(AB + BA)$, while $\langle O\rangle$ is the expectation value of an observable O at $t = 0$.

At time t the particle will be confined, within the same confidence limits, to an interval $\mathcal{I}_t = [\langle x_t\rangle - l_t, \langle x_t\rangle + l_t]$, where $\langle x_t\rangle$ is the expectation value of position at time t, if l_t is positive number defined by

$$(\Delta x/l)^2 = (\Delta x_t/l_t)^2. \tag{2.7}$$

$(l/\Delta x)$ is a dilation factor whose size depends on our confidence limits. If we denote it by f and use (2.6) we can rewrite (2.7) as

$$l_t^2 = f^2[(\Delta x)^2 + 2\,\mathrm{cov}(x, v)t + (\Delta v)^2 t^2]. \tag{2.8}$$

At time t (2.5) shows that $P(x \not\in \mathcal{I}_t) \leqslant (\Delta x_t/l_t)^2 = (\Delta x/l)^2 = 1/f^2$, so that $100/f^2$ is the percentage confidence limit at all times. Clearly there is no point in choosing f to be less than or equal to one, and so we shall assume that $f > 1$.

The argument which leads to the inequality (2.5) is quite general, and applies to any random variable. Hence if $l'>0$,

$$(\Delta v/l')^2 \geqslant P(v \notin [\langle v\rangle - l', \langle v\rangle + l']). \tag{2.9}$$

If we put $l' = f\Delta v$ in (2.9) we obtain

$$1/f^2 \geqslant P(v \notin [\langle v\rangle - f\Delta v, \langle v\rangle + f\Delta v]). \tag{2.10}$$

Thus $100/f^2$ is also the percentage confidence limit that the velocity lies between $\langle v\rangle - f\Delta v$ and $\langle v\rangle + f\Delta v$.

3. Arrival and departure times

It can be shown (Farina 1977, 1982) that, whether the motion is classical or quantum mechanical, the covariance of position and velocity at time t, which we denote by $\mathrm{cov}_t(x, v)$, is given by

$$\mathrm{cov}_t(x, v) = \mathrm{cov}(x, v) + (\Delta v)^2 t. \tag{3.1}$$

From (3.1) we deduce that position and velocity are uncorrelated at the time $t = -\mathrm{cov}(x, v)(\Delta v)^{-2}$. It will be convenient from now on to take this as the origin of time, when $\mathrm{cov}(x, v) = 0$ and (2.6), (2.8) simplify to

$$(\Delta x_t)^2 = (\Delta x)^2 + (\Delta v)^2 t^2 \tag{3.2}$$

$$l_t^2 = f^2[(\Delta x)^2 + (\Delta v)^2 t^2] \tag{3.3}$$

respectively. The time $t=0$ at which position and momentum are uncorrelated is the time when the standard deviation of position, and length of $\mathscr{I}_t (=2l_t)$, are minimal.

Differentiation of (3.3) with respect to t yields

$$\dot{l}_t = f^2(\Delta v)^2 t/l_t = f(\Delta v)^2 t/[(\Delta x)^2 + (\Delta v)^2 t^2]^{1/2}. \tag{3.4}$$

It is easy to see from (3.4) that $\dot{l}_t$ increases monotonically with time from $-f\Delta v$ when $t \sim -\infty$ through zero when $t=0$ to $+f\Delta v$ when $t \sim +\infty$. Since $\mathscr{I}_t = [\langle x_t\rangle - l_t, \langle x_t\rangle + l_t]$ and $\langle x_t\rangle = \langle x\rangle + \langle v\rangle t$, $\langle v\rangle - \dot{l}_t$ and $\langle v\rangle + \dot{l}_t$ are the velocities of the left and right-hand ends P, Q respectively of $\mathscr{I}_t$. It follows that both of these ends move always in the direction of $\langle v\rangle$ if $|\langle v\rangle| > f\Delta v$. In this case every point of the real line is passed by each end precisely once (provided we extrapolate back in time with the source removed).

The point Q coincides with A when $\langle x_t\rangle + l_t = a$. The point P coincides with A when $\langle x_t\rangle - l_t = a$. These are equivalent to $\langle x_t\rangle - a = \pm l_t$, which in turn is equivalent to the equation $(\langle x_t\rangle - a)^2 = l_t^2$; that is, to $(\langle x\rangle + \langle v\rangle t - a)^2 = l_t^2$. (3.3) shows that this last equation is equivalent to

$$[\langle v\rangle^2 - f^2(\Delta v)^2]t^2 - 2d\langle v\rangle t + d^2 - f^2(\Delta x)^2 = 0 \tag{3.5}$$

where $d = \langle x\rangle - a$.

Equation (3.5) is a quadratic equation for the time t. Its real roots (if any) are the times when P or Q coincides with A. If it has no real roots neither P nor Q ever pass A. (For example, if $\langle v\rangle = \langle x\rangle = a = 0$ when $\mathscr{I}_t$ is the interval $[-l_t, +l_t]$ which always contains A.) The quantity d is the distance of the centre of the position distribution from A when $t=0$.

Equation (3.5) yields two solutions t_1 and t_2 for t. They are given by

$$t_{1,2} = d\langle v\rangle/[\langle v\rangle^2 - f^2(\Delta v)^2] \pm \tfrac{1}{2}\Delta t \tag{3.6}$$

where

$$\Delta t = 2f\left(\frac{(\Delta v)^2 d^2}{[\langle v\rangle^2 - f^2(\Delta v)^2]^2} + \frac{(\Delta x)^2}{\langle v\rangle^2 - f^2(\Delta v)^2}\right)^{1/2}. \tag{3.7}$$

If $|\langle v\rangle| > f\Delta v$ the roots are real and distinct, as independently established above. The choice of the negative sign in (3.6) for t_1 and the positive sign for t_2, then ensures that $t_1 < t_2$.

In the case $|\langle v\rangle| > f\Delta v$, therefore, t_1 and t_2 are the arrival and departure times of $\mathscr{I}_t$ at A. The transit time is the time taken for $\mathscr{I}_t$ to cross A, and so equals $t_2 - t_1 = \Delta t$ by (3.6). When $|\langle v\rangle| \to f\Delta v +$ (3.7) shows that the transit time $\Delta t \to +\infty$.

Suppose now that $|\langle v\rangle| < f\Delta v$. Since $\dot{l}_t$ increases monotonically from $-f\Delta v$ when $t \sim -\infty$ to $+f\Delta v$ when $t \sim +\infty$, the velocity $\langle v\rangle + \dot{l}_t$ of Q is initially negative, but increases monotonically and becomes positive for some value of t. Similarly the velocity $\langle v\rangle - \dot{l}_t$ of P is initially positive, but decreases monotonically with t and becomes negative for some value of t. It follows that when $t \to \pm\infty$ $\mathscr{I}_t$ expands to fill the whole real line, and so no point can be completely traversed. The concepts of arrival time, departure time, and transit time in the sense we are using them collapse in this case.

An example in classical mechanics is a crowd of runners as in a marathon. From (2.10), if $|\langle v\rangle| > f\Delta v$ all but an insignificant number of runners (at most) are going in the direction of $\langle v\rangle \neq 0$, and virtually the whole crowd must cross every point completely. If $|\langle v\rangle| < f\Delta v$ a significant number of runners are moving to the left, and a significant number of runners are moving to the right, so the crowd never completely crosses any point.

4. Lower bound on the transit time

The discussion in § 3 is equally valid whether the motion is governed by classical mechanics or quantum mechanics. What feature, then, distinguishes classical motion from quantum motion? We describe such a feature in this section. It illustrates the so-called time-energy uncertainty principle $\Delta E \cdot \Delta t \gtrsim \hbar$, but in a precise form for this particular case.

The transit time Δt is given by (3.7). Classically there is no limit, in principle, to the precision with which the velocity and position can simultaneously be known. Hence in (3.7) Δv and Δx can be arbitrarily small, and there is no positive lower bound on Δt.

Now consider the quantum case. For Δt to be defined we require $|\langle v\rangle| > f\Delta v$. In this case (3.7) shows that

$$\Delta t \geqslant 2f\Delta x/|\langle v\rangle|. \tag{4.1}$$

The Mandelstam–Tamm inequality (Mandelstam and Tamm 1945, see also McWeeny 1972, Bhattacharyya 1983) states that, for any observable A which does not depend explicitly on the time, if the Hamiltonian also does not explicitly depend on the time then

$$\Delta A_t/|\mathrm{d}\langle A_t\rangle/\mathrm{d}t| \geqslant \hbar/2\Delta E; \tag{4.2}$$

in (4.2) ΔA_t is the standard deviation of A at time t, and ΔE is the (constant) standard deviation in energy (assuming all of these quantities exist, of course). If we put $A = x$ in (4.2) and remember that $\langle x_t \rangle = \langle x \rangle + \langle v \rangle t$ we obtain

$$\Delta x_t / |\langle v \rangle| \geq \hbar / 2\Delta E. \tag{4.3}$$

The inequality (4.3) is true for all times, and in particular if $t = 0$ when $\Delta x_t = \Delta x$. Hence from (4.1)

$$\Delta t \geq f\hbar / \Delta E. \tag{4.4}$$

Since $f > 1$ we deduce from (4.4) that

$$\Delta t > \hbar / \Delta E. \tag{4.5}$$

The inequality (4.5) yields a lower bound on the transit time. In fact since $f \gg 1$ for good confidence limits we actually have $\Delta t \gg \hbar / \Delta E$. The positive lower bound on Δt provided is essentially quantum mechanical. It is a particular form of the time–energy uncertainty principle.

5. Conclusion

We have examined the case of a particle moving freely along the x axis when our information is limited to the means and standard deviations of position and velocity. We have shown that, whether the particle is governed by classical mechanics or quantum mechanics, there is an interval $\mathcal{I}_t$ to which the particle is confined to within any pre-assigned percentage confidence limit $100/f^2$. If $|\langle v \rangle| > f\Delta v$ both ends of $\mathcal{I}_t$ travel in the same direction, and there is a well defined arrival time t_1 and departure time t_2 for $\mathcal{I}_t$ at any point, and hence a well defined transit time $\Delta t = t_2 - t_1$; but if $|\langle v \rangle| \leq f\Delta v$, t_1 and t_2 are undefined. The feature which distinguishes the quantum case from the classical case is the inequality (4.5), which is a particular form of the time–energy uncertainty principle.

References

Allcock G R 1979a *Ann. Phys., NY* **53** 253–85

—— 1969b *Ann. Phys., NY* **53** 286–310

—— 1969c *Ann. Phys., NY* **53** 311–48

Bhattacharyya K 1983 *J. Phys. A: Math. Gen.* **16** 2993–6

Davies E B 1976 *Quantum Theory of Open Systems* (London: Academic)

Farina J E G 1977 *Am. J. Phys.* **45** 1200–2

—— 1982 *Int. J. Theor. Phys.* **21** 83–103

Mandelstam L and Tamm I G 1945 *J. Phys. USSR* **9** 249–54

McWeeny R 1972 *Quantum Mechanics: Principles and Formalism* (*vol* 1 *of Topic* 2 *of the International Encyclopedia of Physical Chemistry and Chemical Physics*) (Oxford: Pergamon) pp 84–6

Misra B and Sudarshan E C G 1977 *J. Math. Phys.* **18** 756–63

Simon B and Reed M 1979 *Methods of Modern Mathematical Physics, vol* 3 *Scattering Theory* (New York: Academic)